Countable and Uncountable Sets

Class: III B.Sc Maths
Subject:Real Analysis
Subject Code:22SCCMM10

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COUNTABLE SETS

Definition:

Two sets A and B are said to be equivalent if there exists a bijection of from A to B.

Example 1.

Let A = N and B = {2·,4,6...,2n,...}.
Then f:A⟶B defined by f(n) = 2n is a bijection.
Hence A is equivalent to B even though A has actually 'more' elements than B.

Example 2.

N is equivalent to Z.
The function f:N⟶Z defined by

f(n) = { n/2 if n is even, (1-n)/2 if n is odd.

is a bijection.
Hence N is equivalent to Z.

Definition:

A set A is said to be countably infinite if A is equivalent to the set of natural numbers N.

A is said to be countable if it is finite or countably infinite.

Note:

Let A be a countably infinite set.
Then there is a bijection of from N to A.
Let $f(1) = a_1$, $f(2) = a_2$, f(n) = an, ...
Then $A = \{a_1, a_2, ..., a\square, ...\}$.
Thus all the elements of A can be labelled by using the elements of N.

Example: 1
{2, 4, 6, ..., 2n, ...} is a countable set.

Example: 2
Z is countable.

Example 3
Let A = {1/2, 2/3, 3/4, ...}.
The function $f:N \to A$ defined by $f(n) = n/(n+1)$ is a bijection.
Hence A is countable.

Theorem 1.

A subset of a countable set is countable.

PROOF:

Let A be a countable set and let B $\subseteq$ A.

If A or B is finite, then obviously B is countable.

Hence let A and B be both infinite.

Since A is countably infinite, we can write A = $\{a_1, a_2, ..., a_n, ...\}$.

Let a_{n_1} be the first element in A such that $a_{n_1} \in$ B.

Let a_{n_2} be the first element in A which follows a_{n_1} such that $a_{n_2} \in$ B.

Proceeding like this we get B = $\{ a_{n_1}, a_{n_2}, ...\}$.

Thus all the elements of B can be labelled by using the elements of N.

Hence B is countable.

THEOREM 2.

Q^+ is countable.

PROOF:

Take all positive rational numbers whose numerator and denominator add up to 2.

We have only one number namely 1/1.

Next we take all positive rational numbers whose numerator and denominator add up to 3.

We have 1/2 and 2/1.

Next we take all positive rational numbers whose numerator and denominator add up to 4.

We have 3/1, 2/2 and 1/3.

Proceeding like this, we can list all the positive rational numbers together from the beginning omitting those which are already listed.

Thus we obtain the set {1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4, ...}.

This set contains every positive rational number each occurring exactly once.

Thus Q^+ is countable.

THEOREM 3.

Q is countable.

PROOF.

We know that Q^+ is countable.
Let $Q^+ = \{r_1, r_2, ..., r_n, ...\}$.
$\therefore Q = \{0, \pm r_1, \pm r_2, ..., \pm r_n, ...\}$.
Let $f: N \longrightarrow Q$ be defined by $f(1) = 0$, $f(2n) = r_n$ and $f(2n-1) = - r_n$.
Clearly f is a bijection and hence Q is countable.

THEOREM 4.

Product of two countable sets is countable is countable.

PROOF.

We assume that A and B are countably infinite.
Let $A = \{a_1, a_2, ..., a_n, ...\}$; $B = \{b_1, b_2, ..., b_n, ...\}$.
Now define $f: N\times N \longrightarrow A\times B$ by $f(i, j) = (a_i, b_i)$.
We claim that f is a bijection.
Suppose $x = (p, q) \in N\times N$ and $y = (u, v) \in N\times N$.
Now $f(x) = f(y) \Rightarrow (a_p, b_q) = (a_u, b_v)$
$\Rightarrow a_p = a_u, b_q = b_v$
$\Rightarrow p = u$ and $q = v$
$\Rightarrow (p, q) = (u, v)$
$\Rightarrow x = y$.

∴ f is 1-1.
Now, suppose $(a_m, b_n) \in A \times B$.
Then $(m, n) \in N \times N$ and $f(m, n) = (a_m, a_n)$.
∴ f is onto.
Hence f is a bijection.
Hence A×B is equivalent to N×N which is countable.

THEOREM 5:

Let A be a countably infinite set and f be a mapping of A onto a set B. Then B is countable.

PROOF:

Let A be a countably infinite set and $f: A \to B$ be an onto map.
Let $b \in B$.
Since f is onto, there exits at least one pre-image for b.
Choose one element $a \in A$ such that $f(a) = b$.
Now, define $g: B \to A$ by $g(b) = a$.
Clearly g is 1-1.
∴ B is equivalent to a subset of the countable set A.
∴ B is countable.

THEOREM 6:

Countable union of countable sets is countable.

PROOF:

Let $S = \{A_1, A_2, ..., An, ...\}$ be a countable family of countable sets.
Case (i)
Let each A_i be countably infinite.
Let $A_1 = \{a_{11}, a_{12}, ..., a_1n, ...\}$
$A_2 = \{a_{21}, a_{22}, ..., a2n...\}....$
Now we define a map $f: N \times N \to \cup Ai$ by $f(i, j) = aij$.
Clearly f is onto.
Also by theorem , N×N is countably infinite.
Then ∪ Ai is countably infinite.
Case (ii)
Let each A_i be countable.
For each i choose a set B_i such that B_i is a countably infinite set and $A_i \subseteq B_i$
Then $\cup A_i \subseteq \cup B_i$.
Now, $\cup B_i$ is countable (by case i))
$\therefore \cup A_i$ is countable.

SOLVED PROBLEMS.

Any countably infinite set is equivalent to a proper subset of itself.

Solution:

Let A be a countably infinite set.

Hence $A = \{a_1, a_2, \ldots, a_n, \ldots..\}$.

Let $B = \{a_2, a_3, \ldots, a_n, \ldots\}$.

Clearly B is a proper subset of A.

Define a map $f: A \longrightarrow B$ by $f(a_n) = a_{n+1}$

Clearly f is a bijection.

Hence A is equivalent to B.

Any infinite set contains a countably infinite subset.

Soln.

Let A be an infinite set.

Choose any element $a_1 \in A$.

Now, since A is an infinite set, we can choose another element, $a_2 \in A - \{a_1\}$.

Now, suppose we have chosen $a_1, a_2, ..., a_n$ from A.

Since A is infinite, $A - \{a_1, a_2, ..., a_n\}$ is also infinite.

We can choose a_{n+1} from $A - \{a_1, a_2, ..., a_n\}$.

Now, $B = \{a_1, a_2, ..., a_n, a_{n+1}, ...\}$ is a countably infinite subset of A.

Any infinite set is equivalent to a proper subset of itself.

Solution

Let A be an infinite set.

Then, A contains a countably infinite subset $B = \{a_1, a_2, \ldots, a_n, \ldots\}$.

Clearly, $A = (A - B) \cup B$.

Now consider the following subset C of A given by $C = (A - B) \cup \{a_2, a_3, \ldots.\ a_n, \ldots\} = A - \{a_1\}$.

Clearly C is a proper subset of A.

Consider the function $f: A \to C$ defined by $f(x) = x$ if $x \in A - B$ and $f(a_n) = a_{n+1}$

Obviously f is a bijection.

Hence A is equivalent to C.

UNCOUNTABLE SETS

Definition:

A set which is not countable is called an uncountable set.

Theorem 1:

(0, 1) is uncountable.

PROOF:

Every real number in (0, 1) can be written uniquely as a non-terminating decimal $0.a_1a_2...a_n...$ where $0 \leq a_i \leq 9$ for each i subject to the following restriction that any terminating decimal $0.a_1a_2... a_n000...$ is written as $0.a_1a_2a_3....(a_{n-1})999....$, n
For example 0.54 = 0.53999... 1 = 0.999....
Suppose (0, 1) is countable.
Then the elements of (0, 1) can be listed as $\{x_1, x_2, ..., x_n, ...\}$ where $x_1 = 0. a_{11}a_{12}...a_{1n}...$, $x_2 = 0. a_{21}a_{22}...a_{2n}...$, $x_n = 0. a_{n1}a_{n2}.....a_{nn}...$
Now, for each positive integer n, choose an integer b_n such that $0 \leq b_n \leq 9$ and $b_n \neq a_{nn}$.
Let $y = 0.b_1b_2b_3....$
Clearly $y \in (0, 1)$.
Also y is different from each x_i at least in the i-th place,
Hence $y \neq x_i$ for each i which is a contradiction.
∴ (0, 1) is uncountable.

Corollary 1.

Any subset A of R which contains (0, 1) is uncountable.

Proof:

Suppose A is countable.
Then any subset of A is countable.
Hence we get (0, 1) is countable which is a contradiction.
∴ A is uncountable.

Corollary 2.

R is uncountable.

The results follow directly by taking A = R

Corollary 3.

The set ‘s’ of irrational numbers is uncountable.
Proof:

Suppose s is countable.
We know that Q is countable.
∴ $S \cup Q = R$ is countable which is a contradiction
∴ S is uncountable.


thank you